Chiral Perturbation Theory Approach to NN Scattering Problem

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Abstract

It is shown that chiral perturbation theory (in its original form by Weinberg) can describe NN scattering with positive as well as negative effective range. Some issues connected with unnaturally large NN $^1S_0$ scattering length are discussed.

The chiral perturbation theory approach to the low-energy purely pionic processes [1] has been generalised for processes involving an arbitrary number of nucleons [2], [3]. Weinberg pointed out that for the n-nucleon problem the power counting should be used for the "effective potentials" and not for the full amplitudes. The effective potential is defined as a sum of time-ordered perturbation theory diagrams for the T-matrix excluding those with purely nucleonic intermediate states.

The full S-matrix can be obtained by solving a Lippmann-Schwinger equation (or Schröedinger equation) with this effective potential in place of the interaction Hamiltonian, and with *only* n-nucleon intermediate states [2].

Using renormalization points also characterised by momenta of order of external momenta p or less, the ultraviolet divergences that arise in calculations using effective theory are absorbed into the parameters of the Lagrangian. After renormalization, the effective cut-off is of order p [3]. The Lagrangian is rich enough to contain all possible terms which are allowed by assumed symmetries, so all necessary counter-terms are present in the Lagrangian.

There has been much recent interest in the chiral perturbation theory approach to nucleon-nucleon scattering problems. While some papers concern different constructive calculational and conceptual problems the authors of [4, 5, 6, 7, 8] came to the conclusion that Weinberg's approach to EFT has severe problems. These arise in the description of interactions with positive effective range. Moreover they conclude "there is no effective field theory of NN scattering with nucleons alone" and

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the inclusion of pionic degrees of freedom does not solve the problem [8]. Below it is shown that the Maryland group encountered problems because some features of EFT approach were not taken into account consistently. Performing correct renormalization one sees that the above discouraging conclusions are misleading and nothing is wrong with the EFT (chiral perturbation theory) approach to the nucleon-nucleon scattering problem.

After realising that EFT does not suffer from fundamental problems one is still left with the challenging problem [9] that unnaturally large scattering length of $^1S_0$ wave nucleon-nucleon scattering restricts the validity of EFT to very small values of energy (external momenta). The use of the correct renormalization procedure leads one to the natural solution of this problem via exploiting the freedom of the choice of normalization condition.

The effective non-relativistic Lagrangian for very low energy EFT, when the pions are integrated out is given by [9]

$$\mathcal{L} = N^\dagger i\partial_t N - N^\dagger \frac{\Delta}{2M} N - \frac{1}{2} C_S \left(N^\dagger N\right)^2 - \frac{1}{2} C_T \left(N^\dagger \boldsymbol{\sigma} N\right)^2$$
$$- \frac{1}{2} C_2 \left(N^\dagger \Delta N\right) \left(N^\dagger N\right) + h.c. + ... \tag{1}$$

where the nucleonic field N is a two-spinor in spin space and a two-spinor in isotopic spin space and $\boldsymbol{\sigma}$ are the Pauli matrices acting on spin indices. M is the mass of nucleon and the ellipses refer to additional 4-nucleon operators involving two or more derivatives, as well as relativistic corrections to the propagator. C_T and C_S are couplings introduced by Weinberg [2, 3], they are of dimension $(mass)^{-2}$ and C_2 is of the order $(mass)^{-4}$.

The leading contribution to the 2-nucleon potential is

$$V_0\left(\mathbf{p}, \mathbf{p}'\right) = C_S + C_T\left(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\right). \tag{2}$$

In the $^1S_0$ wave it gives:

$$V_0\left(p, p'\right) = C \tag{3}$$

where $C = C_S - 3C_T$. If we define $C_2 \equiv \frac{C}{2\Lambda^2}$ (where Λ is a parameter of dimension of mass) the next to leading order contribution to the 2-nucleon potential in the $^1S_0$ channel takes the form:

$$V_2(p,p') = C_2\left(p^2 + p'^2\right) = C\left(\frac{p^2 + p'^2}{2\Lambda^2}\right) \tag{4}$$

Formally iterating the potential $V_0 + V_2$ using the Lippmann-Schwinger equation one gets for on-shell ($E = p^2/M$) s-wave T-matrix [7]:

$$\frac{1}{T(p)} = \frac{\left(C_2 I_3 - 1\right)^2}{C + C_2^2 I_5 + p^2 C_2\left(2 - C_2 I_3\right)} - I(p), \tag{5}$$

$$I_n = -M \int \frac{d^3k}{(2\pi)^3} k^{n-3}; \quad I(p) = M \int \frac{d^3k}{(2\pi)^3} \frac{1}{p^2 - k^2 + i\eta} = I_1 - \frac{iMp}{4\pi}, \tag{6}$$

where p is the on-shell momentum and I_1, I_3 and I_5 are divergent integrals.

The authors of the paper [8] carried the renormalization a certain distance without specifying a regularization scheme, choosing as renormalised parameters the experimental values of the scattering length a and the effective range r_e. C and C_2 were fixed by demanding that

$$\frac{1}{T(p)} = -\frac{M}{4\pi}\left(-\frac{1}{a} + \frac{1}{2} r_e p^2 + O\left(p^4\right) - ip\right) \tag{7}$$

Expanding (5) in powers of p^2 and comparing with (7) we see that imaginary parts agree and equating coefficients of terms of order 1 and p^2 we get:

$$\frac{M}{4\pi a} = \frac{\left(C_2 I_3 - 1\right)^2}{C + C_2^2 I_5} - I_1 \tag{8}$$

and

$$\frac{M r_e}{8\pi} = \left(\frac{M}{4\pi a} + I_1\right)^2 \left[\frac{1}{\left(C_2 I_3 - 1\right)^2 I_3} - \frac{1}{I_3}\right] \tag{9}$$

Rewritten in terms of a and r_e the scattering amplitude has the form:

$$Re\left(\frac{1}{T(p)}\right) = \frac{M/(4\pi a) - p^2 I_1 A}{1 + p^2 A} \tag{10}$$

with

$$A \equiv \frac{M r_e}{8\pi}\left(\frac{M}{4\pi a} + I_1\right)^{-1} \tag{11}$$

If one uses sharp cut-off to regularize the divergent integrals then after removal of cut-off one gets the finite result ($I_1 \to \infty$, $A \to 0$ and $I_1 A \to \frac{M r_e}{8\pi}$):

$$\frac{1}{T(p)} = -\frac{M}{4\pi}\left[-\frac{1}{a} + \frac{1}{2} r_e p^2 - ip\right] \tag{12}$$

But one can remove regularization only if $r_e \leq 0$. The problem is that one cannot solve C_2 from (9) if r_e is positive and cut-off $l \to \infty$ (one gets a quadratic equation for C_2 which has no real solutions). The fact that one cannot take the cut-off to infinity and still obtain positive effective range is not surprising. Wigner's theorem states that if the potential vanishes beyond range R then for phase shifts we have [11]:

$$\frac{d\delta(p)}{dp} \geq -R + \frac{1}{2p} sin(2\delta(p) + 2pR). \tag{13}$$

From this one can derive [5]:

$$r_e \leq 2\left[R - \frac{R^2}{a} + \frac{R^3}{3a^2}\right] \tag{14}$$

So, Wigner's theorem (derived from physical principles of causality and unitarity) states that the zero range potential cannot describe positive effective range. Consequently as far as the potential $V_0 + V_2$ (with removed cutoff) is zero-range it fails to describe actual nucleon-nucleon scattering with positive effective range. Does it mean that EFT (with removed cut-off) fails to describe NN scattering with positive effective range? The answer is *no*.

One should remember that although (5) was obtained from the quantum mechanics, it was written as an approximate expression of the QFT scattering amplitude. Although in [7] it was shown that one can make (5) finite non-perturbatively, renormalizing only two parameters, one should remember that our problem is an approximation to the effective field theory. Thus to renormalize (5) consistently one should act in accordance with rules of EFT. In particular, one should remove all divergences by subtracting all divergent integrals or taking into account contributions of counter-terms. If one takes the inverse of (5) and expand in C and C_2, one finds that this expansion contains increasing powers of p^2 with divergent coefficients. To make the amplitude finite, one has to include contributions of infinite number of counter-terms with number of derivatives growing up to infinity, or more simply subtract all divergent integrals at some value of external momenta. One could think that this fact makes the theory completely un-predictive, but it does not. Let us

remind that Weinberg's power counting applies to the renormalised quantities, i.e. after inclusion of contributions of counter-terms. The theory possesses the predictive power because we expect that *renormalized* higher dimensional couplings are heavily suppressed.

The non-perturbative finiteness of the above potential model is not a feature of the original field theory. In addition one hardly can expect that the next approximations to the potential lead to non-perturbatively finite results. So the above given renormalization scheme is not the one one should follow. Furthermore the power counting was performed in the renormalized theory, so one can neglect the contributions of higher order terms into physical quantities (like scattering length and effective range) only in this theory. If one is working in terms of regularized integrals and "bare" coupling constants, then one can not neglect the contributions of higher order terms into effective range (or into any other physical quantity) when the cutoff parameter is removed: there are contributions of an infinite number of terms with more and more severe divergences. So, the equations (8)-(9) are not reliable and hence being unable to solve C_2 in (9) in the removed cut-off limit does not mean that EFT is incapable of describing processes with a positive effective range. The only correct way of renormalizing (5) (consistent with QFT) is the approach by subtraction of integrals. Otherwise one should refer to the original effective field theory: introduce all counter-terms and sum all relevant *renormalised* diagrams up. Note that the above non-perturbative expression for amplitude T is nothing else than the sum of infinite number of perturbative diagrams. It is easy to see that if we subtract all integrals in the perturbative expansion of $T(p)$ and after sum these subtracted series we will get the non-perturbative expression for the amplitude (the inverse of (5)) with subtracted integrals. If one takes into account the contributions of all relevant counter-terms, then one will not encounter any problems like the impossibility of removing the regularization for positive effective range.

As far as renormalised amplitude contains contributions of an *infinite* number of counter-terms with an increasing (up to *infinity*) number of derivatives and consequently *it does not correspond to any zero-range potential* the condition (14) can not keep the effective range non-positive.

Below we will proceed by performing subtractions without specifying regularization. Subtracting divergent integrals at $p^2 = -\mu^2$ in (5) we get the following expression:

$$\frac{1}{T(p)} = \frac{1}{C^R(\mu) + 2p^2 C_2^R(\mu)} + \frac{M}{4\pi}\mu + \frac{M}{4\pi} ip$$

$$= -\frac{M}{4\pi}\left\{\frac{1}{-\frac{M}{4\pi}\left(C^R(\mu)+2p^2C_2^R(\mu)\right)}-\mu-ip\right\} \tag{15}$$

where $C^R(\mu)$ and $C_2^R(\mu)$ are renormalised coupling constants.

We fix $C^R(\mu)$ and $C_2^R(\mu)$ by fitting (15) to the effective range expansion (7). For C^R and C_2^R we have:

$$C^R = \frac{4\pi a}{M(1-a\mu)}; \qquad C_2^R = \frac{\pi}{M(1-a\mu)^2}a^2 r_e \tag{16}$$

Note that above we did not use any regularization at all, so if one uses the dimensional or cut-off (or any other) regularization and subtracts all integrals (takes into account the contributions of all counter-terms which are required by QFT approach) one gets the same results.

The expression (15) can describe positive as well as negative effective range.

The expansion parameter in (15) (if expanded in powers of p^2) is:

$$\lambda = \frac{r_e p^2}{2\left(1/a-\mu\right)} \tag{17}$$

If we take $\mu = 0$ then

$$\lambda = \frac{a r_e p^2}{2} \tag{18}$$

As considered in [9], the result (18) is very discouraging from the EFT point of view. From (18) we see that the expansion of $pcot\delta(p) = ip + \frac{4\pi}{M}\frac{1}{T}$ has the radius of convergence $p^2 \sim 1/(ar_0)$. But $a \sim -1/(8MeV)$ for the $^1S_0$ channel and in general a blows up as a bound state (or nearly bound state) approaches threshold.

It is clear from (17) that the above problem has a quite natural and simple solution. One just needs to exploit the freedom of choice of normalization point μ. For large a (17) becomes $\lambda \approx \frac{r_e p^2}{-2\mu}$. If we take $\mu \sim p$ we get $\lambda \sim -r_e p/2$ and the radius of convergence for $^1S_0$ is $\sim 2/r_e \approx 146MeV$.

If we take $\mu = 140MeV$ in (16) we get $C^R = -1/(99MeV)^2$ and $\Lambda^2 \approx (147MeV)^2$. So, we see that $\Lambda \sim m_\pi$ as was expected for the effective theory, where pions were integrated out. So, the effective range expansion works quite well in this problem. For $a \to \infty$ we have $\Lambda^2 = -\frac{r_e}{2\mu} \sim -(140MeV)^2$ if $\mu \sim 140MeV$. Note that although this value of μ is not of the order of external momenta it does not lead to any power counting problems here.

While this paper was in preparation (the abstract was submitted to the workshop) the same solution of the problem of unnaturally large scattering length's was

suggested in [13]. The authors of [13] used dimensional regularization. As far as dimensional regularization discards power-law divergences the expression for scattering amplitude is already finite in dimensional regularization. To exploit the freedom of the choice of normalization point one needs to perform finite subtractions. The authors of [13] called it an 'unusual subtraction scheme'. As was shown above this solution of unnaturally large scattering length is regularization independent and normalization condition used is very typical and quite natural.

Conclusions

One should be careful performing calculations in the EFT approach. If one follows the way described in [2], [3] then one will not encounter the fundamental problems described in [4, 5, 6, 7, 8]. One can introduce regularization into the potential in the effective theory of nucleons alone (this potential is taken up to some order in EFT expansion) and solve Schröedinger equation using this regulated potential. One can fit parameters of regularized theory using the technique of cut-off effective theory. Otherwise, if the regularization is supposed to be removed then it is *necessary* to include contributions of an infinite number of counter-terms, or otherwise subtract all divergences. The amplitude obtained without inclusion of counter-terms evidently satisfies the constraint that the effective range has an upper bound which goes to zero in the removed cut0off limit. One can come to the incorrect conclusion that the theory can not describe positive effective range. The analogous problems appear in the theory with included pionic (and other) degrees of freedom and they should be treated in a similar way.

The problem of unnaturally large scattering length of $^1S_0$ wave can be solved by appropriate choice of normalisation point within conventionally renormalised theory. This solution does not depend on regularization.

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